Math 155, ***Lecture Notes- Bonds*** **Name**____________

Section 9.3 The Integral Test and p-Series

In this section we will study series with positive terms. In particular, in this section we consider the connection between improper integrals, bounded areas over infinite intervals, and plots of partial sums sequences.

> **THEOREM 9.10 The Integral Test**
>
> If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then
>
> $$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$
>
> either both converge or both diverge.

Ex. 1: Apply the Integral Test: $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Let $f(n) = a_n = \frac{1}{n^2}$ and $f(x) = \frac{1}{x^2}$. We need to show that f is positive, continuous, and decreasing for all $x \geq 1$.

We know $1 > 0$ and $x^2 > 0$ for all $x \geq 1$. We can see that $\frac{1}{x^2} > 0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $f(x) > 0$ for all $x \geq 1$.

We know 1 is a polynomial and x^2 is a polynomial. We can see that $f(x) = \frac{1}{x^2}$ is continuous on $[1, \infty)$ since the ratio of

RocFic) of continuous functions is continuous, except where the denominator is zero. $x^2 = 0$ at $x = 0$, but $0 \notin [1, \infty)$.

To see that $f(x)$ is decreasing on $(1, \infty)$, we need to consider $f'(x)$ and show that $f'(x) < 0$ on $(1, \infty)$.

We know $f'(x) = \frac{d}{dx}[x^{-2}]$

$f'(x) = -2x^{-3}$

$f'(x) = \frac{-2}{x^3}$

More Ex. 1:

Find critical numbers

(A) $f'(x) = 0$

$-\frac{2}{x^3} = 0$

False!

NONE

&

(B) $f'(x)$ is undefined

Now $\frac{-2}{x^3}$ is undefined

when $x^3 = 0$,

at $x = 0$

Test $f'(x)$ with $x = 2$, since $2 \in (1, \infty)$:

$f'(2) = \frac{-2}{(2)^3}$

$f'(2) = -\frac{1}{4}$

$f'(2) < 0$. This means that $f(x)$ is decreasing for $x \geq 1$.

Consider $\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^2}\,dx$

$= \lim_{b \to \infty} \int_1^b x^{-2}\,dx$

$= \lim_{b \to \infty} \left[-1 \cdot x^{-1}\right]_1^b$

$= -\lim_{b \to \infty} \left[\frac{1}{b} - 1\right]$

$= -[0 - 1]$

$= 1$ Since $\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx$ converges, the Integral test tells us that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as well. $\square$

Ex. 2: Apply the Integral Test: $\sum_{n=1}^{\infty} \frac{1}{n}$

Let $f(n) = a_n = \frac{1}{n}$ and $f(x) = \frac{1}{x}$. We need to show that f is positive, continuous, and decreasing for all $x \geq 1$.

We know $1 > 0$ and $x > 0$ for all $x \geq 1$. We can see that $\frac{1}{x} > 0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $f(x) > 0$ for all $x \geq 1$.

We know 1 is a continuous polynomial and x is a continuous polynomial. We can see that $f(x) = \frac{1}{x}$ is continuous on $[1, \infty)$ since the ratio of (RoCF(c)) continuous functions is continuous, except where the denominator is zero. The denominator is zero at $x = 0$, but $0 \notin (1, \infty)$.

To see that $f(x)$ is decreasing on $(1, \infty)$, we need to consider $f'(x)$ and show that $f'(x) < 0$ on $(1, \infty)$.

We know $f'(x) = \frac{d}{dx}[x^{-1}]$

$$f'(x) = -1 \cdot x^{-2}$$

$$f'(x) = \frac{-1}{x^2}$$

Find critical numbers:

(A) $f'(x) = 0$

$\frac{-1}{x^2} = 0$

False!

None

(B) $f'(x)$ is undefined

$\frac{-1}{x^2}$ is undefined when $x^2 = 0$, at $x = 0$.

Test $f'(x)$ with $x = 2$, since $2 \in (1, \infty)$

$$f'(2) = \frac{-1}{(2)^2}$$

$$f'(2) = -\frac{1}{4}$$

$f'(2) < 0$. This means that $f(x)$ is decreasing for $x \geq 1$.

More Ex. 2:

Consider $\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b$$

$$= \lim_{b\to\infty} \big[\ln(b) - \ln(1)\big]$$

$$= \infty - 0$$

$$= \infty$$ ← This means the improper integral diverges.

Since $\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx$ diverges, the Integral Test tells us that $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n}$ diverges as well. $\square$

p-Series and Harmonic Series

There are two types of series that we should be able to recognize by inspection and we should be able to determine the convergence, or divergence of these series very quickly. We need to be able to do this in order to use this information for "comparison" purposes. In the next section we begin to understand series behavior by making use of comparing particular series to the known behavior of "simpler" series. But first, we need a few definitions.

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

the harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ ← Diverges

the general harmonic series: $\sum_{n=1}^{\infty} \frac{1}{an+b}$ ← Diverges

THEOREM 8.5 A Special Type of Improper Integral

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$$

THEOREM 9.11 Convergence of *p*-Series

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

1. converges if $p > 1$, and
2. diverges if $0 < p \le 1$.

Ex. 3: Determine convergence: $\sum_{n=1}^{\infty} \frac{2}{\sqrt[5]{n^2}} = \sum_{n=1}^{\infty} \frac{2}{n^{2/5}}$

$$= 2\sum_{n=1}^{\infty} \frac{1}{n^{2/5}}$$

This series is a p-series with $p = \frac{2}{5}$. Since $0 < p \leq 1$, this p-series diverges.

Ex. 4: Determine convergence: $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}+\cdots$

$$\underset{n=1}{1} + \underset{n=2}{\frac{1}{2\sqrt{2}}} + \underset{n=3}{\frac{1}{3\sqrt{3}}} + \underset{n=4}{\frac{1}{4\sqrt{4}}} + \underset{n=5}{\frac{1}{5\sqrt{5}}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/2}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This series is a p-series with $p = \frac{3}{2}$. Since $p \geq 1$, this p-series converges.

Ex. 5: Determine convergence: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let $f(n) = a_n = \frac{1}{n^2+1}$ and $f(x) = \frac{1}{x^2+1}$. We need to show that f is positive, continuous, and decreasing for all $x \geq 1$.

We know $1 > 0$ and $x^2+1 > 0$ for all $x \geq 1$. We can see that $\frac{1}{x^2+1} > 0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $f(x) > 0$ for all $x \geq 1$.

We know 1 is a continuous polynomial and x^2+1 is a continuous polynomial. We can see that $f(x) = \frac{1}{x^2+1}$ is continuous on $[1,\infty)$ since the ratio of continuous functions is continuous, (RoCFiC), except where the denominator is zero. x^2+1 is never zero.

To see that $f(x)$ is decreasing on $(1,\infty)$, we need to consider $f'(x)$ and show that $f'(x) < 0$ on $(1,\infty)$.

We know $f'(x) = \frac{d}{dx}\left[(x^2+1)^{-1}\right]$

$$f'(x) = -1 \cdot (x^2+1)^2 \cdot (2x)$$

$$f'(x) = \frac{-2x}{(x^2+1)^2}$$

Find critical numbers:

Ⓐ $f'(x) = 0$	Ⓑ $f'(x)$ is undefined
$\frac{-2x}{(x^2+1)^2} = 0$	$\frac{-2x}{(x^2+1)^2}$ is undefined
$-2x = 0$	$x^2+1 = 0$
$x = 0$	False!, None

Test $f'(x)$ with $x = 2$, since $2 \in (1,\infty)$

$$f'(2) = \frac{-2(2)}{[(2)^2+1]^2}$$

$$f'(2) = \frac{-4}{25}$$

$f'(2) < 0$. This means $f(x)$ is decreasing for $x \geq 1$.

More Ex. 5:

Consider $\displaystyle\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^2+1}\,dx$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \Big[\arctan(x)\Big]_1^b$$

$$= \lim_{b\to\infty} \left[\arctan(b) - \arctan(1)\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} \qquad \star\star\star$$

$$= \frac{\pi}{4}$$ ← This means the improper integral converges.

Since $\displaystyle\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^2+1}\,dx$ converges, the Integral Test tells us that $\displaystyle\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n^2+1}$ converges as well. □

$\star\star\star$

Let $\arctan(1) = \theta$

$\tan(\theta) = 1$

$\theta = \frac{\pi}{4}$ ✓

Ex. 6: Determine convergence: $\sum_{n=1}^{\infty} ne^{\frac{-n}{2}}$

Let $f(n)=a_n=ne^{\frac{-n}{2}}$ and $f(x)=xe^{-\frac{x}{2}}$. We need to show that f is positive, continuous, and decreasing for all $x\geq 1$. We can write $f(x)=xe^{\frac{-x}{2}}=\frac{x}{e^{\frac{x}{2}}}$. We know $x>0$ and $e^{\frac{x}{2}}>0$ for all $x\geq 1$. We can see that $\frac{x}{e^{\frac{x}{2}}}>0$ since a ratio of positive numbers is positive. (RoPNiP). This means that $f(x)>0$ for all $x\geq 1$.

We know x is a continuous polynomial and $e^{\frac{x}{2}}$ is a continuous function. We can see that $f(x)=\frac{x}{e^{\frac{x}{2}}}$ is continuous on $[1,\infty)$ since the ratio of continuous functions is continuous, (RocFic), except where the denominator is zero. $e^{\frac{x}{2}}$ is never zero.

To see that $f(x)$ is decreasing on $(1,\infty)$, we need to consider $f'(x)$ and show that $f'(x)<0$ on $(1,\infty)$.

We know $f'(x)=\frac{d}{dx}[xe^{\frac{-x}{2}}]$

$$f'(x)=(x)\cdot(e^{\frac{-x}{2}})\cdot(-\tfrac{1}{2})+(e^{\frac{-x}{2}})\cdot(1)$$

$$f'(x)=e^{\frac{-x}{2}}(-\tfrac{x}{2}+1)$$

Find critical numbers:

(A) $f'(x)=0$	(B) $f'(x)$ is undefined
$e^{\frac{-x}{2}}(-\frac{x}{2}+1)=0$	$e^{\frac{-x}{2}}(-\frac{x}{2}+1)=0$
$-\frac{x}{2}+1=0$	$e^{\frac{x}{2}}=0$
$1=\frac{x}{2}$	False ∇
$x=2$	None

Test $f'(x)$ with $x=3$, since $3\in(2,\infty)$

$$f'(3)=e^{\frac{-3}{2}}(\tfrac{-3}{2}+1)$$

$$f'(3)=\frac{-\frac{1}{2}}{e^{3/2}}$$

$f'(3)<0$. This means $f(x)$ is decreasing for $x\geq 2$.

More Ex. 6:

Consider $\int_1^\infty f(x)\,dx = \int_1^\infty x e^{-\frac{x}{2}}\,dx$

$$= \lim_{b\to\infty} \int_1^b x e^{-\frac{x}{2}}\,dx$$

$$= \lim_{b\to\infty} \left[(x)(-2e^{-\frac{x}{2}})\Big|_1^b - \int_1^b (-2e^{-\frac{x}{2}})(dx) \right]$$

$$= \lim_{b\to\infty} \left[(-2xe^{-\frac{x}{2}})\Big|_1^b + 2\int_1^b e^{-\frac{x}{2}}\,dx \right]$$

$$= \lim_{b\to\infty} \left(\left[-2xe^{-\frac{x}{2}}\right]_1^b + 2\left[-2e^{-\frac{x}{2}}\right]_1^b \right)$$

$$= \lim_{b\to\infty} \left[-2xe^{-\frac{x}{2}} - 4e^{-\frac{x}{2}}\right]_1^b$$

$$= -2\cdot\lim_{b\to\infty} \left[e^{-\frac{x}{2}}(x+2)\right]_1^b$$

$$= -2\lim_{b\to\infty} \left[e^{-\frac{b}{2}}(b+2) - e^{-\frac{1}{2}}(1+2)\right]$$

$$= -2\lim_{b\to\infty} \frac{b+2}{e^{\frac{b}{2}}} + 2e^{-\frac{1}{2}}(3)$$

$$= \frac{6}{\sqrt{e}} - 2\lim_{b\to\infty} \frac{\frac{d}{db}(b+2)}{\frac{d}{db}(e^{\frac{b}{2}})}$$

$$= \frac{6}{\sqrt{e}} - 2\lim_{b\to\infty} \frac{1}{\frac{1}{2}e^{b/2}}$$

$$= \frac{6}{\sqrt{e}} - 2\cdot 0$$

$$= \frac{6}{\sqrt{e}}$$

← This means the improper integral converges.

Since $\int_1^\infty f(x)\,dx = \int_1^\infty xe^{-\frac{x}{2}}\,dx$ converges, the Integral Test tells us that $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty ne^{-\frac{n}{2}}$ converges as well. ∎

Use "Parts"

$\int u\,dv = uv - \int v\,du$

Let $u = x$

$\frac{du}{dx} = 1$

$du = dx$

$dv = e^{-\frac{x}{2}}dx$

$v = \int e^{-\frac{x}{2}}dx$

$v = \int e^{z}(-2dz)$

$v = -2e^{z} + c$

$v = -2e^{-\frac{x}{2}}$

Let $z = -\frac{x}{2}$

$\frac{dz}{dx} = -\frac{1}{2}$

$-2dz = dx$

Indeterminate Form stop!

$\lim_{b\to\infty} \frac{b+2}{e^{b/2}} \to \boxed{\frac{\infty}{\infty}}$

Use L'Hôpital's Rule